1. $f(x) = \cot x$

$\cot x = 1/\tan x = \frac{\cos x}{\sin x}$

$\frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{d}{dx} f(x).$

By quotient rule $\frac{d}{dx}\frac{u}{v} = \frac{u'v - v'u}{v^2}.$

$\therefore \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{-\sin x[\sin x] - \cos x \cos x}{\sin^2 x}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x.} = -\frac{\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$

$= -1 - \frac{\cos^2 x}{\sin^2 x} = -1 - \cot^2 x$

but $\csc^2 x = 1 + \cot^2 x$

therefore $\frac{d}{dx} f(x) = -\csc^2 x$

2. $f(t) = (7 + t^7)\left(\frac{3}{\sqrt[3]{t}} + 3t\right).$

$f(t) = (7 + t^7 (3t^{-3} + 3t)$

Expanding brackets

$f(t) = 21t^{-3} + 21t + 3t^4 + 3t^8$

$= 21(t + t^{-3}) + 3t^4 + 3t^8$

$= 21(1 + -3t^{-4}) + 12t^3 + 24t^7$

$\frac{d}{dt} f(t) = 21 - 63t^{-4} + 12t^3 + 24t^7.$

$= 21 - \frac{63}{\sqrt[4]{t}} + 12t^3 + 24t^7$

3. $r(\theta) = \frac{\sin\theta}{1 + \csc\theta}$ but $\csc\theta = \frac{1}{\sin\theta}$

$r(\theta) = \frac{\sin\theta}{1 + \frac{1}{\sin\theta}}$ Simplifying the denominator $1 + \frac{1}{\sin\theta} = \frac{\sin\theta + 1}{\sin\theta}$

$$r(\theta) = \frac{\sin\theta}{\frac{\sin\theta + 1}{\sin\theta}} = \frac{\sin\theta \times \sin\theta}{\sin\theta + 1} = \frac{\sin^2\theta}{\sin\theta + 1}$$

but $\sin^2\theta = 1 - \cos^2\theta$

$$r(\theta) = \frac{1-\cos^2\theta}{\sin\theta + 1}$$

By quotient rule

$$\frac{d}{d\theta}(r(\theta)) = \frac{\frac{d}{d\theta}(1-\cos^2\theta)(\sin\theta+1) - \frac{d}{d\theta}(\sin\theta+1)(1-\cos^2\theta)}{(\sin\theta+1)^2}$$

but $\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$

$$\frac{d}{d\theta}(r(\theta)) = \frac{\frac{d}{d\theta}\left[\left(1-\frac{1}{2}(1+\cos 2\theta\right)\right](\sin\theta+1) - \frac{d}{d\theta}\left[\sin\theta+1\right]\left(1-\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)}{(1+\sin\theta)^2}$$

$$= \frac{\frac{d}{d\theta}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right](\sin\theta+1) - \frac{d}{d\theta}\left[\sin\theta+1\right]\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta\right]}{(1+\sin\theta)2.}$$

$$\frac{d}{d\theta}(r(\theta)) = \frac{\left[-\sin\theta\cos 2\theta\right](\sin\theta+1) - \cos\theta\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta\right]}{(1+\sin\theta)^2}$$

opening brackets

$$\frac{d}{d\theta}(r(\theta)) = \frac{-\sin^2\theta\cos 2\theta - \cos 2\theta\sin\theta - \frac{1}{2}\left[\cos\theta(\cos 2\theta\right]}{(1+\sin\theta)^2}$$

$$= \frac{\sin^2\theta\cos\theta + 2\sin\theta\cos\theta}{(1+\sin\theta)2.}$$

4. $g(x) = 2x \sin x \qquad x = 3\pi/2$

$g(x) = (2x)(\sin(x)] = uv$

$\frac{d}{dx}(g(x)) = u'v + v'u$

$= 2 \sin x + 2x \cos x$

at $x = \frac{3\pi}{2} = 2 \sin 270 + 2(270) \cos 270$

$= -2$

5. $f(x)$

$f(1) = 5$

$f'(1) = 4$

$h(x) = \frac{f(x)}{x+1}$

$f(x) = (x+1)h(x)$

$f(1) = (1+1)\, h(1) = 5$

$\Rightarrow h(1) = 5/2$

$h(x) = \frac{f(x)}{x+1}$

$h(1) = \frac{f(1)}{1+1} = \frac{5}{2}.$

$h'(1) = \frac{f'(1)}{x+1} = \frac{4}{2} = 2.$

$y = f(x) \;, y = h(x)$

At $x = 1.$

$(y-5) = 4(x-1)$

$(y-y_0) = m(x-x_0)$

Gradient of the tangent line to $h(x)$ $m = \cancel{h}f'(1)x^{(-1)} = \frac{1}{\cancel{2}}x^{-1} = -1/\cancel{x} = -1/2$

At $x = 1$

$h(x) = h(1) = 5/2$

$(y - 5/2) = -1/2(x-1)$

$y - 5/2 = -\frac{x}{2} + \frac{1}{2}$

$y = -\frac{x}{2} + 3$

6. $f(t) = 4t^3 - \frac{3}{t^5} = 4t^3 - 3t^{-5}$

$y^{(n)} = \frac{-1^{(n-1)}(n-1)!\,a^n}{(ax+b)^n}$

$f'(t) = 12t^2 + 15t^{-6}$

$f''(t) = 24t - 90t^{-7}$

$f''t = 24 + 630t^{-8} = 24 + \frac{630}{t^8}$

7. 42nd and 55th derivative of $f(x) = \cos x$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = +\sin x$

$f^{iv}(x) = \cos x$

$f^n(x) = \sin x$ if n is odd and multiple of 3

$f^n(x) = -\sin x$ if n is odd and not a multiple of 3

$f^n(x) = \cos x$ if n is even and multiple of 4

$f^n(x) = -\cos x$ if n is ~~odd~~ even and not a multiple of 4

Now 42 is a multiple of 3

(i) $f^{42}(x) = \sin x$

(ii) 55^{th} is not a multiple of 3 and 4 and it is odd

$f^{55}(x) = -\sin x$

8. $\frac{d}{dx}(-2 f(x)) \Big|_{x=5}$

Now

$\frac{d}{dx}[-2 f(x)]$

$= -2 \frac{d}{dx} f(x) \Big|_{x=5} = -2 f'(5) = -2 \times -2 = 4$

(b) $\frac{d}{dx}(x^2 f(x)] \Big|_{x=3}$

Now $\frac{d}{dx}(x^2 f(x)) = x^2 f'x + f(x) \frac{d}{dx} x^2$

At $x = 3$ $= 3^2 f'(3) + 2 \times 3 + (3)$

$= 9 \times 7 + 6 \times 3 = 81$

(c) $y = \frac{g(x)}{f(x)}$ at $x = 4$

First find the value of the given function at $x = 4$

$y \Big|_{x=4} = \frac{g(4)}{f(4)} = -1/4$

The equation of the tangent line at given point (x_0, y_0) is given as $y - y_1 = m(x - x_1)$

where m = slope of the tangent line

Now $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{gx}{f(x)}\right. = \frac{f(x) g'(x) - g(x) f'(x)}{[f(x)]^2}$

$m = \frac{f(4) g'(4) - g(4) (f'(4))}{(f(4))^2}$

$m = \frac{4 \times 1 - (-1)(-1)}{(-1)^2} = \frac{3}{1}$

$m = 3$

Now eqn of the tangent line at point $(4, -\frac{1}{4})$ is

$$y - -\frac{1}{4} = 3(x-4)$$

$$y = 3x - 12 - \frac{1}{4} = \underline{\underline{3x - \frac{49}{4}}}$$